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A single server queueing model with self-generation of priorities, customer induced interruption and retrial of customers

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Abstract: In this article we consider a single server queueing model in which customers arrive according to Poisson process. At the time of arrival, all customers are classified as ordinary. If the server is busy the arriving customers enter an orbit of infinite capacity. Each customer in the orbit tries, independently of each other, to access the server at a constant rate σ. Each customer in the orbit, independently of others, generate priority with inter-occurrence time exponentially distributed with parameter γ. A priority generated customer is immediately taken for service if the server is free. Else, such a customer is placed in a waiting space A_1 of capacity 1 which is reserved only for priority generated customers. We consider a customer induced interruption while service is going on. The interruption occurs according to a Poisson process. The interrupted customers will enter into a buffer B_1 of finite capacity K and they will spend a random period of time for completion of interruption. The duration of the interruption of customers in B_1 follows an exponential distribution. The service facility consists of one server and the duration of service times of ordinary, priority and interruption completed customers follows an exponential distribution with appropriate representations. Various performance measures are obtained for an appropriate system designing.

Index Terms: Customer induced interruption; Level Dependant Quasi-Birth-Death Process; Matrix Analytic Method; Retrial Queues; Self-generation of priorities.

1 INTRODUCTION

Queueing models with self-generation of priorities, customer induced interruption and retrial of customers has wide applications. Some applications in health care systems are mentioned in articles **[3]**, **[16]**, **[4]**,**[11]** and in books **[14]**, **[7]**, **[15].** The pioneer works on self-generation of priorities are **[5]**,**[9]**,**[8]**. There is a survey on queues with interruption **[10]** and customer induced interruption appeared in **[6]**. Retrial queues are extensively investigated in **[18]**, **[2]**, **[1]**. The present paper discusses a single server queueing system in which the arrival of customers form a Poisson process and service times are exponentially distributed. Using the matrix-analytic method (refer **[12]** for details) we study this model and some measures of system performance in the steady state are derived and analyzed graphically.

2 MODEL DESCRIPTION

We consider a single server queueing system in which customers arrive according to Poisson process with parameter λ. An arriving customer enters service immediately if the server is free and if the server is busy, the customer enters an orbit of infinite capacity. Each customer in the orbit independently tries to access the server according to Poisson process with parameter σ. A retrial customer who finds the server busy returns to the orbit with probability δ (< 1) and leaves the system forever with probability $1-\delta$. The customers in the orbit can generate priority according to Poisson process with parameter γ. A priority generated customer is immediately taken for service if the server is free, otherwise, move to a waiting space A_1 of capacity one which is reserved only for priority generated customers. If the waiting space A_1 is occupied with a previous priority generated customer, the new priority generated customer will leave the system forever. We also consider a customer induced interruption when service is going on and the interruption occurs according to a Poisson process with parameter θ. The interrupted customers will enter into a buffer B_1 of finite capacity K if there is space available and the customer will be lost forever if no space for the customer in buffer B_1. The service provided here is non-preemptive and the service times are exponentially distributed with parameters μ_1,μ_2 and μ_3 for ordinary, priority generated and interruption completed customers respectively. When an interruption occurs, the customer currently in service will be forced to leave the service facility and the freed server is ready to offer services to other customers. The interrupted customer will spend a random period of time for completion of interruption and it follows an exponential distribution with parameter η and customers who completed the interruption will move to a buffer B_2 whose size is also K. We assume that priority generated customers will never undergo interruption and not more than one interruption is allowed for a customer during service.

Also assume that the sum of a number of customers in B_1 and B_2 should be less than or equal to K. Otherwise, if buffer B_2 is full, we cannot accommodate one more interruption-completed customer from the buffer B_1.

The model is studied as a Quasi Birth-Death (QBD) process and *Matrix Geometric solution* is obtained. The following notations are used to describe the model:

$N_1(t) -$ Number of customers in the orbit at time t,
$N_2(t) -$ Number of busy servers at time t.

$$S(t) - \begin{cases} 1, & \text{server busy with ordinary customers} \\ 2, & \text{server busy with priority generated customers} \\ 3, & \text{server busy with interruption completed customers} \end{cases}$$

$N_3(t) -$ Number of priority generated customers waiting for service at time t.
$N_4(t) -$ Number of interruption completed customer in buffer B_2 at time t.
$N_5(t) -$ Number of interrupted customers in buffer B_1 at time t.

Under the assumptions on arrival and service processes $\{\chi(t): t \geq 0\}$ where $\chi(t) = \{N_1(t), N_2(t), S(t), N_3(t), N_4(t), N_5(t)\}$ form a continuous time Markov chain on the state space $\Omega = L_1(i) \cup L_2(i)$, where $L_1(i) = \{(i, 0, w, b_2, b_1) : i \geq 0;\ w = 0,1;\ b_2, b_1 = 0,1,2, \cdots, K;\ b_2 + b_1 \leq K\}$, $L_2(i) = \{(i, 1, s, w, b_2, b_1): i \geq 0,\ s = 1,2,3;\ w = 0,1;\ b_2, b_1 = 0,1,2, \cdots, K;\ b_2 + b_1 \leq K\}$.

By partitioning the state space into levels with respect to the number of customers in the orbit, the generator of above Markov process is of the form:

$$Q = \begin{bmatrix} A_{10} & A_0 & . & . & . & . & . & . \\ A_{21} & A_{11} & A_0 & . & . & . & . & . \\ . & A_{22} & A_{12} & A_0 & . & . & . & . \\ & & \ddots & \ddots & \ddots & & & \\ . & . & . & A_{2N} & A_{1N} & A_0 & . & . \\ . & . & . & . & A_{2N+1} & A_{1N+1} & A_0 & . \\ . & . & . & . & . & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_0, A_{10}, A_{2i}, A_{1i},\ \ i = 1,2,3, \cdots$ are square matrices of order $4(K+1)(K+2)$ and defined as follows:

Matrix A_{10} has the following transitions:

Entries of A_{10} are the transitions within level 0.
Let $s = 1,2,3$; $w = 0,1$; $b_2, b_1 = 1,2,3, \ldots K$; $b_2 + b_1 \leq K$.

- $(0,0, w, b_2, b_1) \xrightarrow{-\lambda - b_1\eta} (0,0, w, b_2, b_1)$; if $b_1 = K,\ b_2 + b_1 = K,\ b_2 + b_1 < K$
- $(0,0, w, b_2, b_1) \xrightarrow{-\lambda} (0,0, w, b_2, b_1)$; otherwise
- $(0,1, s, w, b_2, b_1) \xrightarrow{-\lambda - b_1\eta} (0,1, s, w, b_2, b_1)$; if $b_1 = K,\ b_2 + b_1 = K,\ b_2 + b_1 < K$
- $(0,1, s, w, b_2, b_1) \xrightarrow{-\lambda} (0,1, s, w, b_2, b_1)$; otherwise

Matrix A_0 has the following transitions:

Entries of A_0 are the transitions from i to $(i+1)$. Let $s = 1,2,3$; $w = 0,1$; $b_2, b_1 = 1,2,3, \ldots K$; $b_2 + b_1 \leq K$.

- $(i, 0, w, b_2, b_1) \xrightarrow{\lambda} (i + 1,0, w, b_2, b_1)$;
- $(i, 1, s, w, b_2, b_1) \xrightarrow{\lambda} (i + 1,1, s, w, b_2, b_1)$;

Matrix A_{2i} has the following transitions:

Entries of A_{2i} are the transitions from i to $(i-1)$. Let $s = 1,2,3$; $b_2, b_1 = 1,2,3, \ldots K$; $b_2 + b_1 \leq K$.

- $(i, 0,0, b_2, b_1) \xrightarrow{i\gamma} (i - 1,0,1, b_2, b_1)$;
- $(i, 1, s, 0, b_2, b_1) \xrightarrow{i\gamma} (i - 1,1, s, 1, b_2, b_1)$;
- $(i, 0,1, b_2, b_1) \xrightarrow{i\gamma} (i - 1,0,1, b_2, b_1)$;
- $(i, 1, s, 1, b_2, b_1) \xrightarrow{i\gamma} (i - 1,1, s, 1, b_2, b_1)$, if $b_1 \neq K$.
- $(i, 1, s, 1, b_2, b_1) \xrightarrow{i\gamma + \theta} (i - 1,1, s, 1, b_2, b_1)$, if $b_1 = K$ *or* $b_2 + b_1 = K$;
- $(i, 1, s, 0, b_2, b_1) \xrightarrow{\theta} (i - 1,1, s, 0, b_2, b_1 + 1)$
- $(i, 1, s, 0, b_2, b_1) \xrightarrow{\theta} (i - 1,1, s, 0, b_2, b_1)$, if $b_1 = K$ or $b_2 + b_1 = K$;
- $(i, 0, w, b_2, b_1) \xrightarrow{i\sigma} (i - 1,1,1, w, b_2, b_1)$, for $w = 0,1$;
- $(i, 1, s, w, b_2, b_1) \xrightarrow{i\sigma(1-\delta)} (i - 1,1, s, w, b_2, b_1)$, for $w = 0,1$;
- $(i, 1,1,0, b_2, b_1) \xrightarrow{\mu_1} (i - 1,0,1,0, b_2, b_1)$;
- $(i, 1, s, 0, b_2, b_1) \xrightarrow{\mu_s} (i - 1,0, s, 0, b_2, b_1)$, if $s = 2,3$ *and* $b_2 = 0$;
- $(i, 1, s, 0, b_2, b_1) \xrightarrow{\mu_s} (i - 1,0,3,0, b_2, b_1)$, if $s = 2,3$ *and* $b_2 \neq 0$;
- $(i, 1, s, 1, b_2, b_1) \xrightarrow{\mu_s} (i - 1,0,2,0, b_2, b_1)$, if $s = 1,2,3$.

Matrix A_{1i} has the following transitions:

Entries of A_{1i} are transitions within level i. Let $b = 0,1; w = 0,1; b_2, b_1 = 1,2,3, \ldots K; b_2 + b_1 \leq K$;

- $(i, 0, w, b_2, b_1) \xrightarrow{-\lambda - i\gamma - i\sigma - b_1\eta} (i, 0, w, b_2, b_1)$, if $b_1 = K$ or $b_1 + b_2 = K$ or $b_1 + b_2 < K$;
- $(i, 0, w, b_2, b_1) \xrightarrow{-\lambda - i\gamma - i\sigma} (i, 0, w, b_2, b_1)$, otherwise .
- $(i, 1, s, w, b_2, b_1) \xrightarrow{\Delta} (i, 1, s, w, b_2, b_1)$ if $b_1 = K$ or $b_1 + b_2 = K$ or $b_1 + b_2 < K$; where $\Delta = -\lambda - \mu_s - i\gamma - i\sigma(1-\delta) - \theta - b_1\eta$.
- $(i, 1, s, w, b_2, b_1 \xrightarrow{\nabla} (i, 1, s, w, b_2, b_1)$,otherwise; where $\nabla = -\lambda - \mu_s - i\gamma - i\sigma(1-\delta) - \theta$

3 STEADY STATE ANALYSIS

The Model described here is a level dependent Quasi Birth Death (QBD) process. So, for further analysis we use an algorithmic solution based on Neuts-Rao **[13]** truncation method. When we apply this method our process χ transforms to $\bar{\chi}$ with the infinitesimal generator :

$$\bar{Q}=\begin{bmatrix} A_{10} & A_0 & . & . & . & . & . & . & . \\ A_{21} & A_{11} & A_0 & . & . & . & . & . & . \\ . & A_{22} & A_{12} & A_0 & . & . & . & . & . \\ & & \ddots & \ddots & \ddots & & & & \\ . & . & . & A_{2N-1} & A_{1N-1} & A_0 & . & . & . \\ . & . & . & . & A_2 & A_1 & A_0 & . & . \\ . & . & . & . & . & A_2 & A_1 & A_0 & . \\ . & . & . & . & . & . & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_1=A_{1N}$ and $A_2=A_{2N}$

Let the steady state probability vector of the Markov process be $x=(x_0,x_1,x_2,\dots)$. We take

$$x_{N+i}=x_{N-1}R_N^{i+1}, \qquad i=0,1,2,\dots \qquad (1)$$

where R_N is the minimal solution of the matrix quadratic equation $R_N^2A_{2N}+R_NA_{1N}+A_0=0$.

4 STEADY STATE VECTOR

Again, $x\bar{Q}=0$ leads to

$$x_{N-i}=x_{N-i-1}R_{N-i};\ \ i=1,2,\dots N-2 \qquad (2)$$

and

$$x_1=x_0R_1;\ \ i=1,2,\dots N-2 \qquad (3)$$

where $R_{N-i}=-A_0(A_{1N-i}+R_{N-i+1}A_{2N-i+1})^{-1}$ and $R_1=-A_0(A_{11}+R_2A_{22})^{-1}$. Finally from $x_0A_{10}+x_1A_{21}=0$ we find x_0 as the steady state distribution of finite state Markov chain with generator $A_{10}+R_1A_{21}$. From $(1),(2),(3)$, we get x_i for $i=1,2,3,\dots$. Now x is calculated by dividing each x_i with the normalizing constant $\sum_{i=0}^{\infty}x_ie$.

5 SYSTEM STABILITY

Theorem: The system under discussion is stable.

Proof: Consider the *Lyapunov* test function defined by $\phi(s)=i$ where s is a state in level i. For a state in level i, the mean drift y_s is given by

$$y_s=\sum_{p\neq s}[\phi(p)-\phi(s)]q_{sp}$$
$$=\sum_{s'}[\phi(s')-\phi(s)]q_{ss'}+\sum_{s''}[\phi(s'')-\phi(s)]q_{ss''}+\sum_{s'''}[\phi(s''')-\phi(s)]q_{ss'''}$$

where s',s'',s''' vary over states belonging to levels $i-1,\ i$ and $i+1$ respectively.

Then $\phi(s)=i,\ \phi(s')=i-1,\ \phi(s'')=i,\phi(s''')=i+1$. Now, $y_s=-\sum_{s'}q_{ss'}+\sum_{s'''}q_{ss'''}$.

Since $(1-\delta)>0$, for any $\epsilon>0$, we can find K^* large enough such that $y_s<-\epsilon$ for any s belonging to level $i\geq K^*$. Hence the theorem follows from Tweedie's **[17]** result.

6 PERFORMANCE MEASURES

Let $\xi=(\xi_0,\xi_1,\xi_2,\cdots)$ be our steady-state probability vector of the Markov process χ. For the evaluation of system performance measures, we partition each ξ_i, $i\geq 0$ as follows: Let $\xi_i=(w_i,x_i,y_i,z_i)$ where each vector corresponds to the probability that the server is functioning with i customers in the orbit.

- The probability that the server is idle, $pidle=\sum_{i=0}^{\infty}w_ie$
- The probability that the server is busy with an ordinary customer, $psbor=\sum_{i=0}^{\infty}x_ie$
- The probability that the server is busy with priority generated customer, $psbpr=\sum_{i=0}^{\infty}y_ie$
- The probability that the server is busy with interruption completed customers in B_2, $psbb2=\sum_{i=0}^{\infty}z_ie$
- The probability that the server is idle with customers in the orbit, $psidco=\sum_{i=0}^{\infty}w_ie-w_0e$
- Expected number of customers in the orbit, $exor=\sum_{i=1}^{\infty}i\xi_ie$
- Expected number of customers in the orbit when the server is idle, $exsidle=\sum_{i=1}^{\infty}iw_ie$
- Expected number of customers in the orbit when the server is busy with an ordinary customer, $exsbor=\sum_{i=1}^{\infty}ix_ie$
- Expected number of customers in the orbit when the server is busy with priority generated customer, $exsbpr=\sum_{i=1}^{\infty}iy_ie$
- Expected no. of customers in the orbit when the server is busy with interruption completed customer, $exsbb2=\sum_{i=1}^{\infty}iz_ie$
- Successful retrial rate, $surerate=\sigma\sum_{i=1}^{\infty}iw_ie$
- Overall retrial rate, $ovrerate=\sigma\sum_{i=1}^{\infty}i\xi_ie$
- The fraction of success rate of retrial,

$$frsurerate=\frac{surerate}{ovrerate}=\frac{\sum_{i=1}^{\infty}iw_ie}{\sum_{i=1}^{\infty}i\xi_ie}$$

Let $\xi_i=\zeta_i(b,s,w,b_2,b_1)$ where $\zeta_i(b,s,w,b_2,b_1)$ is a row vector corresponding to $N_2(t)=b,\ S(t)=s$, $N_3(t)=w,\ N_4(t)=b_2,\ N_5(t)=b_1$ with $b=0,1$; $s=1,2,3$; $w=0,1$; $b_2,b_1=0,\cdots,K$; $b_2+b_1\leq K$.

- Expected number of priority generated customers lost from the system,

$$exprl=\sum_{i=1}^{\infty}\sum_{b=0}^{1}\sum_{s=1}^{3}\sum_{b_2=0}^{K}\sum_{b_1=0}^{K-b_2}i\zeta_i(b,s,1,b_2,b_1)e$$

- Expected number of interrupted customers lost from the system.

$$exinl=\sum_{i=1}^{\infty}\sum_{b=0}^{1}\sum_{s=1}^{3}\sum_{b_2=0}^{K}i\zeta_i(b,s,1,b_2,K-b_2)e$$

7 NUMERICAL ILLUSTRATION

(a) Effect of changing γ in some performance measures :

Let $\sigma = 0.6, \eta = 3.6, \delta = 0.3, \theta = 2.4, \lambda = 2.2, \mu_1 = 0.4, \mu_2 = 1.9, \mu_3 = 0.3$ and compute the following measures using different γ values and plot Figure 1 and Figure 2. From Figure 1 we observe that when γ increases the expected number of customers in the orbit when the server is busy with priority generated customers decreases. From Figure 2 we observe that when γ increases successful retrial rate decreases gradually and overall retrial rate also decreases. But the fraction of success rate of retrial increases with the increase of γ .

(b) Effect of changing θ in some performance measures :

Let $K = 1,\ \sigma = 0.6,\ \eta = 1.6,\ \gamma = 5,\ \delta = 0.3\ ,\ \lambda = 5.2\ ,\ \mu_1 = 1.5\ ,\ \mu_2 = 2.5\ ,\ \mu_3 = 1.0$ and compute the following measures using different θ values and plot Figure 3. From Figure 3 we observe that when θ increases the expected number of customers in the orbit when the server is busy with interruption completed customers increases initially and reaches a maximum and then decreases.

(c) Effect of changing θ in some performance measures :

Let $\sigma = 0.6,\ \eta = 1.6,\ \gamma = 5,\ \delta = 0.3,\ \lambda = 5.2,\ \mu_1 = 1.5,\ \mu_2 = 2.5,\ \mu_3 = 1.0$ and compute the expected number of interrupted customer lost using different θ values and for K=1,2 plot Figure 4.
From Figure 4 we observe that when θ increases the expected number of interrupted customers lost increases initially and reaches a maximum and then decreases. Also when buffer size K increases, the expected number of interrupted customers lost decreases.

(d) Effect of changing γ and λ in some performance measures :

Let K=2, $\sigma = 0.6\ , \eta = 1.6\ ,\ \theta = 2.4\ ,\ \delta = 0.3\ ,\ \mu_1 = 1.5\ ,\ \mu_2 = 4.0\ ,\ \mu_3 = 1.0$ and compute the expected number of priority generated customers lost using different γ values and plot Figure 5. From Figure 5 we observe that when λ increases the expected number of priority generated customers decreases gradually and when γ increases, the expected number of priority generated customers lost increase very fast for smaller values of λ.

Figure 1

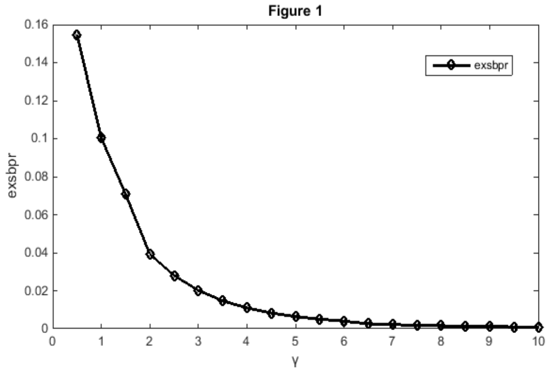


Figure 2

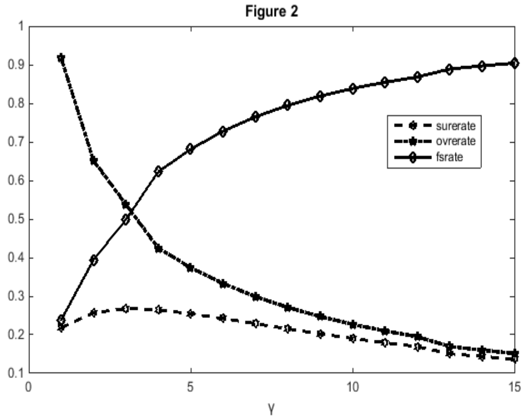


Figure 3

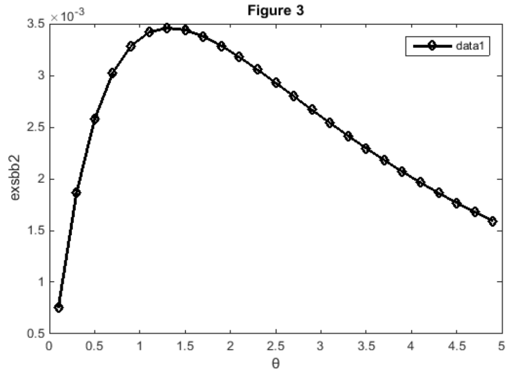
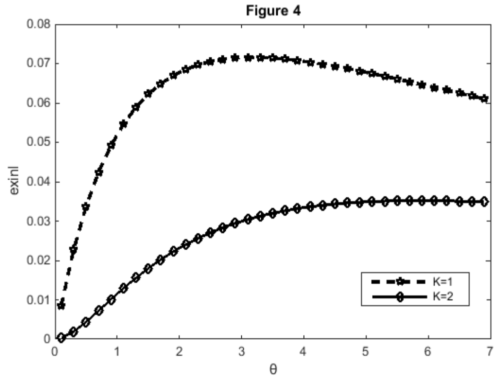
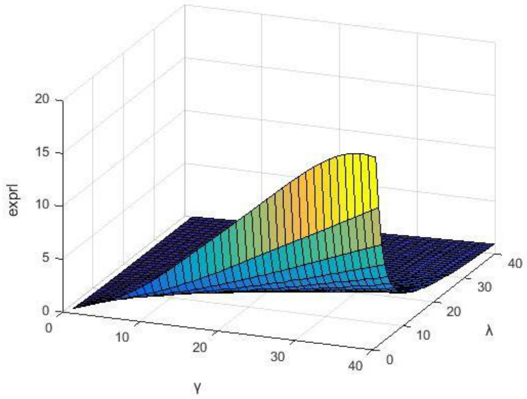


Figure 4





8 CONCLUSION

A single server retrial queueing system with self-generation of priorities is analyzed in this paper. The arrival of customers are according to Poisson process and service times follow different Exponential distributions. The interruption we discussed here is customer induced interruption. Performance measures required for an appropriate system designing were computed and numerically analyzed.

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